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CONJUGATE POINTS AND SIMPLE ZEROS FOR ORDINARY LINEAR DIFFERENTIAL EQUATIONS

BY
THOMAS L. SHERMAN[1]

I. **Introduction.** In a very interesting paper in 1958, Hartman proved the following result [2]:

The equation

$$(1) \qquad L_n y = y^{(n)} + p_{n-1}(x) y^{(n-1)} + \cdots + p_0(x) y = 0$$

with $p_i \in C(a, b)$, has a nontrivial solution with n zeros, each being counted in accordance with its multiplicity, if and only if there is a solution with n distinct zeros on (a, b).

Left as an open question in that paper was the question of whether the result remains true when the restriction that the interval be open is removed. The answer to this question is given in §4 of this paper. In addition there is the problem of giving a more exact description to the solution with n distinct zeros, that is, does there exist a solution with exactly n distinct zeros all of which are simple? Also, can the first zero be specified? These latter two questions are answered in §2, and more specifically by

THEOREM 1. *Suppose there is a solution of* (1) *with a zero at a and n zeros on* $[a, b)$. *Then there is a solution with a simple zero at a whose first n zeros on* $[a, b)$ *are simple zeros.*

In §3 the results of §2 are applied to establish the continuity of the first conjugate point $\eta_1(t)$ of the point t with respect to the coefficients in (1). In [5] it was established that $\eta_1(t)$ is an increasing function of t. The techniques used there could be used to show $\eta_1(t)$ is a continuous function of t, however an easier proof is presented here using the results of §2 (see also [3]).

In §5 is discussed the consequences of the existence of a solution of (1) with n distinct zeros on $[a, \eta_1(a)]$. As is shown in §4, such a solution need not necessarily exist.

The results of this paper are stated in terms of equation (1) but could just as easily be stated for a more general equation as in [6].

2. **Existence of a solution with simple zeros.** To begin with a lemma dealing with the behavior of real valued functions is proven.

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LEMMA 1. *Let $f(x)$ and $g(x)$ be functions with the following properties*:

(i) *$g(x)$ has a zero of order $q \geqq 0$ at $x = a$ and $g^{(q)}(a) > 0$,*

(ii) *$f(x)$ has a zero of order $p > q$ at $x = a$ and $f^{(p)}(a) > 0$,*

(iii) *$f(x)$ and $g(x)$ are positive on $(a, b]$,*

(iv) *$f(x), g(x) \in C^p[a, b]$.*

Then given any $\alpha > 0$ there is a constant c, $0 < c < \alpha$ such that $h(x) = f(x) - cg(x)$ has a simple zero at some point $t_c \in (a, b)$ and $h(x) \neq 0$ for $x \in (t_c, b]$.

Proof. Let $\alpha > 0$ be given. By (i), (ii) and (iv) there is an $\varepsilon > 0$ such that $f^{(p)}(x) > 0$ and $g^{(q)}(x) > 0$ on $[a, a+\varepsilon]$. Let

$$\alpha_1 = \min\left\{\alpha, \min_{x \in [a+\varepsilon, b]} f(x)/2 \max_{x \in [a+\varepsilon, b]} g(x)\right\}.$$

Then for any $c \leqq \alpha_1$, $h_c(x) = f(x) - cg(x) > 0$ for $x \in [a+\varepsilon, b]$. Let t_c be the last zero of $h_c(x)$ in $[a, a+\varepsilon)$, $0 < c \leqq \alpha_1$. It shall now be shown that the map $H_c \colon c \to t_c$ is 1-1 and t_c is an increasing function of c. In fact the latter will obviously imply the former which is itself immediate since $0 < c_1 < c_2$ implies by (iii) $c_1 g(x) < c_2 g(x)$, hence $h_{c_2} < h_{c_1}$ and hence $t_{c_1} < t_{c_2}$. Also $t_c \to a$ as $c \to 0$. Further, since H_c is 1-1, and $(0, \alpha_1]$ is uncountable, so is the image set of elements t_c. Hence there is an infinite number of distinct limit points $\{t_{c_j}\}$; $t_{c_1} > t_{c_2} > \cdots$ converging to a as $j \to \infty$, for which there is a set of image points $t_{c_{ji}} \to t_{c_j}$ as $i \to \infty$ for each $j = 1, 2, \ldots$.

Suppose now the lemma is not true. Then each image point t_c corresponds to a zero of order at least two of h_c and hence a zero of

$$W(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} \quad \text{at } x = t_c.$$

Hence, since W is continuous along with its first $p-1$ derivatives, $W(t_{c_j}) = 0$ and by Rolle's theorem $W^{(k)}(t_{c_j}) = 0$ for $k < p$, $j = 1, 2, \ldots$. It is immediate that $c_j = f(t_{c_j})/g(t_{c_j}) = f'(t_{c_j})/g'(t_{c_j})$. Note that $h'_{c_j}(t_{c_j}) = 0$ and $f'(t_{c_j}) > 0$ implies $g'(t_{c_j}) \neq 0$. Suppose now that it has been shown that $c_j = f(t_{c_j})/g(t_{c_j}) = \cdots = f^{(k)}(t_{c_j})/g^{(k)}(t_{c_j})$, and $g^{(k)}(t_{c_j}) \neq 0$ where $0 < k < p$. Hence

$$\begin{vmatrix} f^{(r)}(t_{c_j}) & g^{(r)}(t_{c_j}) \\ f^{(s)}(t_{c_j}) & g^{(s)}(t_{c_j}) \end{vmatrix} = 0, \qquad 0 \leqq r < s \leqq k < p.$$

Now

$$W^{(k)}(x) = \begin{vmatrix} f(x) & g(x) \\ f^{(k+1)}(x) & g^{(k+1)}(x) \end{vmatrix} + (k-1) \begin{vmatrix} f'(x) & g'(x) \\ f^{(k)}(x) & g^{(k)}(x) \end{vmatrix} + \cdots.$$

Evaluating $W^{(k)}(x)$ at $x = t_{c_j}$ this becomes

$$W^{(k)}(t_{c_j}) = 0 = \begin{vmatrix} f(t_{c_j}) & g(t_{c_j}) \\ f^{(k+1)}(t_{c_j}) & g^{(k+1)}(t_{c_j}) \end{vmatrix}.$$

Hence $f(t_{c_j})g^{(k+1)}(t_{c_j})-f^{(k+1)}(t_{c_j})g(t_{c_j})=0$. If $g^{(k+1)}(t_{c_j})=0$ then $f^{(k+1)}(t_{c_j})=0$ but $k+1\leqq p$ hence $f^{(k+1)}(x)>0$ for $x\in(a, a+\varepsilon]$, hence $g^{(k+1)}(t_{c_j})\neq 0$. Thus

$$f(t_{c_j})-\frac{f^{(k+1)}(t_{c_j})}{g^{(k+1)}(t_{c_j})}g(t_{c_j}) = 0 \quad \text{and} \quad c_j = \frac{f^{(k+1)}(t_{c_j})}{g^{(k+1)}(t_{c_j})} \quad \text{for all } k < p.$$

Hence h_{c_j} has a zero of order $p+1$ at t_{c_j}. Now letting $j\to\infty$, $h_{c_j}\to h_0$ also has a zero of order $p+1$ at $x=a$ which is impossible since $h_0(x)=f(x)$(2).

REMARK. A similar result holds if the roles of a and b are interchanged.

A series of seven lemmas are now proven which lead up to Theorem 1. Some of the results presented as lemmas are of interest in their own right.

To avoid repetition, it shall be understood that by the term solution is meant nontrivial solution.

For completeness the following definition is included.

DEFINITION 1. The first conjugate point $\eta_1(a)$ of the point a is the smallest number $b>a$ for which there exists a nontrivial solution of (1) which vanishes at a and has n zeros, counting multiplicities in $[a, b]$.

It has been proven [5] that if there is a solution with n zeros on $[a, \infty)$ then $\eta_1(a)$ exists.

LEMMA 2. *If $n\geqq 3$ and there is a solution of (1) with n zeros on $[a, b)$ then there is a solution of (1) with n zeros on $[a, b)$ which vanishes at a and has at least one zero in $(a, \eta_1(a))$.*

Proof. Suppose this were not the case. Let $\phi(x)$ be a solution of (1) with a zero of order k_1, $(1\leqq k_1<n)$, at a and a zero of order $k_2\geqq n-k_1$ at $\eta_1(a)<b$. It is known that for some k_1 such a solution exists [5]. It can further be supposed that $\phi(x)>0$ on $(a, \eta_1(a))$.

Case 1. $k_2=1$. Let $\Psi(x)$ be a solution of (1) with zeros at a and $(a+\eta_1(a))/2$ of orders k_1-1 and 1 respectively. By supposition Ψ has no other zeros on $[a, b)$. $\Psi(x)$ can be chosen positive in $(a, (a+\eta_1(a))/2)$. Hence there is a point s_1, $\eta_1(a)<s_1<b$ at which Ψ and ϕ have the same sign and hence so do ϕ and $\alpha\Psi$ for any $\alpha>0$. Also for any $\alpha>0$, $\alpha\Psi$ crosses ϕ at least once in $(a, (a+\eta_1(a))/2)$. Pick $\alpha=\phi(s_1)/\Psi(s_1)$ then $z=\phi-\alpha\Psi$ has zeros at a and s_1 of orders k_1-1 and 1 and a zero in $(a, (a+\eta_1(a))/2)$. A total of n zeros in (a, b) and a zero in $(a, \eta_1(a))$ which is a contradiction.

Case 2. $k_2>1$. Let $\Psi(x)$ have a zero at $\eta_1(a)$ of order $k_2-2\geqq 0$, at $(a+\eta_1(a))/2$ of order one, and at a of order $n-(k_2-2)-2\leqq k_1$. Note that $n-(k_2-2)-2>0$. Ψ has been assigned $n-1$ zeros, hence by assumption it has no other zeros on $[a, b)$. It can be assumed that $\alpha\phi\Psi>0$ on some interval $[(s_1, \eta_1(a))\cup(\eta_1(a), s_2)]$ where $(a+\eta_1(a))/2<s_1<\eta_1(a)<s_2<b$, and α is any positive constant. Let

$$\alpha = \min\,(\phi(s_1)/\Psi(s_1),\ \phi(s_2)/\Psi(s_2))$$

(2) The author would like to thank Grant Gustafson for the proof given here of Lemma 1.

then $z=\phi-\alpha\Psi$ has a zero at a of order $n-(k_2-2)-2>0$, at $\eta_1(a)$ of order k_2-2; a zero in $[s_1, \eta_1(a))$ of order at least one and in $(\eta_1(a), s_2]$ a zero of order at least one. A total of at least n zeros with a zero in $(a, \eta_1(a))$ and a zero at a.

LEMMA 3. *Suppose there is a solution ϕ of* (1) *with n zeros on $[a, b)$, a zero at a, and $R-1$ distinct zeros on $[a, \eta_1(a))$. Suppose also no solution of* (1) *with a zero at a and n zeros on $[a, b)$ has more than $R-1$ distinct zeros on $[a, \eta_1(a))$. Then the zeros of ϕ in $(a, \eta_1(a))$ are simple zeros.*

Proof. Suppose ϕ has $R+p$ distinct zeros, $p\geqq 0$, at the points $a=t_1<t_2<\cdots<t_{R+p}$ where $t_{R-1}<\eta_1(a)\leqq t_R$ and $t_{R+p}<b$, of multiplicities $m_1, m_2, \ldots, m_{R+p}$ where $\sum_{i<R+p} m_i<n\leqq\sum_{i\leqq R+p} m_i$ and ϕ has no other zeros on $[t_1, t_{R+p}]$. Suppose the lemma is not true. Then there is a j, $1<j<R$, such that $m_j>1$.

Let $\Psi(x)$ be a nontrivial solution of (1) such that Ψ has a zero at t_j of order m_j-2 and Ψ has a zero at $(t_{j-1}+t_j)/2$. Assign $n-m_j$ more zeros at $t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_{R+p}$ by assigning zeros at the t_i $(i\neq j, i<R+p)$ of orders m_i, then assign zeros at t_{R+p} until a total of $n-1$ zeros have been assigned to Ψ in $[t_1, t_{R+p}]$.

Since $\Psi(x)$ has at least $R-1$ distinct zeros in $[a, \eta_1(a))$ including a zero at a and a total of $n-1$ zeros in $[a, b)$, Ψ cannot have another distinct zero in $[a, \eta_1(a))$ nor can it vanish at t_j with a zero of order greater than m_j-2, hence ϕ and Ψ have no zeros in common on $[a, t_{R+p})$ other than those assigned above.

Let $\gamma=(t_{j-1}+t_j)/2$ and let $\delta=\min\{t_{j+1}, \eta_1(a)\}$. Since $\phi(x)$ and $-\Psi(x)$ are both solutions and the zeros of ϕ and Ψ at t_j are both even or both odd it can be assumed that $\phi(x)\Psi(x)>0$ for $x\in(\gamma, t_j)\cup(t_j, \delta)$ and $\Psi(x)>0$ for $x\in(t_j, \delta)$ (hence $\phi(x)>0$ for $x\in(t_j, \delta)$). Now since $\gamma>t_{j-1}$ and $\phi^{(m_j-2)}(t_j)=0<\Psi^{(m_j-2)}(t_j)$, for any $\alpha>0$, $\alpha\Psi^{(m_j-2)}(t_j)>\phi^{(m_j-2)}(t_j)$ and $\alpha\Psi(\gamma)=0<|\phi(\gamma)|$. Hence for any positive α there is a $x_\alpha\in(\gamma, t_j)$ such that $\alpha\Psi(x_\alpha)=\phi(x_\alpha)$. Let $\alpha=\phi((t_j+\delta)/2)/\Psi((t_j+\delta)/2)>0$ and $z(x)=\phi(x)-\alpha\Psi(x)$. Then $z(x)$ is a solution of (1) with zeros at $t_1, t_2, \ldots, x_\alpha, t_j, (t_j+\delta)/2, t_{j+1}, \ldots, t_{R+p}$ and a total of n zeros in $[t_1, t_{R+p}]$. But $z(x)$ then has R distinct zeros in $[t_1, \eta_1(a))$, contradicting the maximality of R.

LEMMA 4. *Suppose there is a solution of* (1) *with n zeros on $[a, b)$, a zero at a, and $R-1$ distinct zeros on $[a, \eta_1(a))$. Suppose also no solution of* (1) *with a zero at a and n zeros on $[a, b)$ has more than $R-1$ distinct zeros on $[a, \eta_1(a))$. Then there is a solution of* (1) *with a simple zero at a, $R-1$ distinct zeros in $[a, \eta_1(a))$, and n zeros on $[a, b)$.*

Proof. By Lemma 3 any solution with n zeros on $[a, b)$, a zero at a, and $R-1$ zeros on $[a, \eta_1(a))$ has only simple zeros in $(a, \eta_1(a))$. Let $S=\{m \mid$ there is a nontrivial solution $\phi(x)$ of (1) with n zeros on $[a, b)$, $R-1$ zeros on $[a, \eta_1(a))$ and a zero of order $m>0$ at $x=a\}$. Let $m_1=\min m\in S$. Suppose, if possible $m_1>1$. Let $\phi(x)$ be a solution of (1) having zeros at $a=t_1<t_2<\cdots<t_{R-1}<t_R<\cdots<t_{R+p}$, where $t_{R-1}<\eta_1(a)\leqq t_R$ and $t_{R+p}<b$, of orders m_1, $m_2=1, \ldots, m_{R-1}=1$, $m_R, \ldots, m_{R+p}$, where $\sum_{i<R+p} m_i<n\leqq\sum_{i\leqq R+p} m_i$. Let $\Psi(x)$ be a solution of (1) having zeros at t_1,

$(t_1+t_2)/2, t_2, \ldots, t_{R-1}, t_R, \ldots, t_{R+p}$ of orders $m_1-1, 1, m_2, \ldots, m_{R-1}, m_R, \ldots, n-\sum_{i<R+p} m_i - 1$. Ψ has been assigned $n-1$ zeros and can have no others by the maximality of R. There are two cases.

Case 1. $m_{R+p}-(n-\sum_{i<R+p} m_i-1)>0$ is even. Then there is an ε, $0<\varepsilon<b-t_{R+p}$ such that $|\Psi(x)|>|\phi(x)|>0$ for $x\in[t_{R+p}-\varepsilon, t_{R+p})\cup(t_{R+p}, t_{R+p}+\varepsilon]$. Since both $-\phi$ and $-\Psi$ are solutions it can be supposed that $\Psi(x)\phi(x)>0$ in this set. Let

$$\alpha = \min\{\phi(t_{R+p}-\varepsilon)/\Psi(t_{R+p}-\varepsilon), \phi(t_{R+p}+\varepsilon)/\Psi(t_{R+p}+\varepsilon)\}.$$

Then $\alpha\Psi(x)$ crosses $\phi(x)$ at least once in $[t_{R+p}-\varepsilon, t_{R+p})$ and at least once in $(t_{R+p}, t_{R+p}+\varepsilon]$. Let these points of intersection be $\bar{t}_1$ and $\bar{t}_2$. Then $z(x)=\phi(x)-\alpha\Psi(x)$ is a solution of (1) with zeros at $t_1, t_2, \ldots, t_{R-1}, t_R, \ldots, \bar{t}_1, t_{R+p}, \bar{t}_2$ of orders $m_1-1, p_2, \ldots, p_{R-1}, p_R, \ldots, \bar{p}_1, n-\sum_{i<R+p} m_i-1, \bar{p}_2$ where $p_i\geqq m_i$ and $\bar{p}_1\geqq 1$, $\bar{p}_2\geqq 1$. $z(x)$ has a total of at least n zeros in $[a, b)$, $R-1$ distinct zeros in $[a, \eta_1(a))$ and a zero at a of order m_1-1 contradicting the minimality of m_1.

Case 2. $m_{R+p}-(n-\sum_{i<R+p} m_i-1)>0$ is odd. Since $-\phi$ and $-\Psi$ are solutions it can be assumed that $\phi(x)>0$ and $\Psi(x)>0$ for $x\in(t_1, (t_1+t_2)/2)$. Since $m_1-1<m_1$ and $\phi((t_1+t_2)/2)>\Psi((t_1+t_2)/2)=0$, it follows that, for any $\alpha>0$, $\alpha\Psi(x)=\phi(x)$ for some $x_\alpha\in(t_1, (t_1+t_2)/2)$. Since $m_{R+p}-(n-\sum_{i<R+p} m_i-1)$ is odd there is an ε, $0<\varepsilon<b-t_{R+p}$ such that $\phi(x)\Psi(x)>0$ for $x\in(t_{R+p}, t_{R+p}+\varepsilon]$ and $|\Psi(x)|>|\phi(x)|$ in that interval. This follows since both ϕ and Ψ have $\sum_{i<R+p} m_i$ zeros in $[a, t_{R+p})$, $\sum_{1<i<R+p} m_i$ zeros in common in $[t_2, t_{R+p})$, both start out with the same sign, ϕ has one more zero at $x=a$ and Ψ has a zero of order one at $x=(t_1+t_2)/2$ where as ϕ does not vanish there. Let $\alpha=\phi(t_{R+p}+\varepsilon)/\Psi(t_{R+p}+\varepsilon)$ then $z=\phi-\alpha\Psi$ has zeros at $t_1, x_\alpha, t_2, \ldots, t_{R+p-1}, t_{R+p}, t_{R+p}+\varepsilon$ of orders at least $m_1-1, 1, m_2, \ldots, m_{R+p-1}, n-\sum_{i<R+p} m_i-1, 1$. A total of n zeros in $[a, b)$, a zero at a, and R distinct zeros in $[a, \eta_1(a))$, contradicting the choice of R.

LEMMA 5. *Let the following hypothesis be satisfied*:

(i) *There is a solution* $\phi(x)$ *of* (1) *with n zeros on* $[a, b)$, *a simple zero at a*, $R-1$ *distinct zeros on* $[a, \eta_1(a))$, *and its first n zeros occur at* $R+p$ *distinct points.*

(ii) *No solution of* (1) *with a zero at a and n zeros on* $[a, b)$ *has more than* $R-1$ *distinct zeros on* $[a, \eta_1(a))$.

(iii) *No solution of* (1) *with a simple zero at a, n zeros on* $[a, b)$ *and* $R-1$ *distinct zeros on* $[a, \eta_1(a))$ *has its first n zeros occurring at more than* $R+p$ *points.*

(iv) *Let the first n zeros of* ϕ *occur at points* $a=t_1<t_2<\cdots<t_{R+p}$ *of orders* $m_1=1, m_2=1, \ldots, m_{R-1}=1, m_R, \ldots, m_{R+p}$ (*by Lemma* 3) *where* $\sum_{i<R+p} m_i<n\leqq\sum_{i\leqq R+p} m_i$.

Then, $m_R=m_{R+1}=\cdots=m_{R+p-1}=1$, *if* $p\geqq 1$.

Proof. Suppose there is a j, $R\leqq j<R+p$ such that $m_j>1$. Let j be the smallest such integer.

Let $\Psi(x)$ be a solution of (1) such that $\Psi(x)$ has a zero at t_j of order m_j-2 and $\Psi(x)$ has a zero of order one at $(t_1+t_2)/2$. Assign $n-1-(m_j-1)$ more zeros at

$t_1, t_2, \ldots, t_{j-1}, t_{j+1}, \ldots, t_{R+p}$ by assigning zeros at the t_i $(i \neq j, i < R+p)$ orders m_i, then assign zeros at t_{R+p} until a total of $n-1$ zeros have been assigned to Ψ in $[t_1, t_{R+p}]$. This can be done since

$$\sum_{i \leqq R+p; i \neq j} m_i + m_j - 2 + 1 = \sum_{i \leqq R+p} m_i - 1 \geqq n-1 > \sum_{i < R+p} m_i - 1$$
$$= \sum_{i < R+p; i \neq j} m_i + m_j - 2 + 1.$$

$\Psi(x)$ can have no other zeros in $[a, b)$ by the choice of R (note that $j \geqq R$).

Since both $-\phi$ and $-\Psi$ are solutions it can be assumed that $\phi(x)\Psi(x) > 0$ for $x \in (t_{j-1}, t_j) \cup (t_j, t_{j+1})$. Since ϕ and Ψ have the same order zero at $t=a$ and, by the minimality of j, Ψ has one more zero than ϕ on $[a, t_j)$, namely the zero at $(t_1+t_2)/2$, it follows that $\phi^{(m_1)}(a)\Psi^{(m_1)}(a) < 0$. Also for any constant $\alpha > 0$, there is an $\varepsilon_\alpha > 0$ such that $\alpha|\Psi(x)| > |\phi(x)| > 0$ for $x \in [t_j - \varepsilon_\alpha, t_j) \cup (t_j, t_j + \varepsilon_\alpha]$ where $t_{j-1} < t_j - \varepsilon_\alpha < t_j < t_j + \varepsilon_\alpha < t_{j+1}$.

Let

$$\alpha = \min\left\{\frac{\phi(t_j-\varepsilon_1)}{2\Psi(t_j-\varepsilon_1)}, \frac{\phi(t_j+\varepsilon_1)}{2\Psi(t_j+\varepsilon_1)}, \left|\frac{\phi^{(m_i)}(t_i)}{2\Psi^{(m_i)}(t_i)}\right| i \neq j, \tfrac{1}{2}\right\}$$

and let $z_\alpha = \phi - \alpha\Psi$. Then z_α has a zero in $(t_j - \varepsilon_1, t_j)$ and a zero in $(t_j, t_j + \varepsilon_1)$. Further z_α has zeros at $t_1, t_2, \ldots, t_{R+p}$ and a total of n zeros in $[t_1, t_{R+p}]$ since ϕ and Ψ had $n-2$ zeros in common. Since $z_\beta = \phi - \beta\Psi$ has the same properties for $0 < \beta \leqq \alpha$, it must only be established that for some β in this range, z_β has no other zeros in $[t_1, t_{R+p}]$.

By the way α was chosen, there exist $\delta_1 > 0, \delta_2 > 0, \ldots, \delta_{R+p} > 0$ such that

$$|\phi(x)| > \beta|\Psi(x)|$$
$$\text{for } x \in (t_1, t_1+\delta_1] \cup \left\{\bigcup_{i=2; i \neq j}^{R+p-1} ([t_i - \delta_i, t_i) \cup (t_i, t_i + \delta_i])\right\} \cup [t_{R+p} - \delta_{R+p}, t_{R+p})$$

for $0 < \beta \leqq \alpha$. Also $|\phi(x)| > 0$ on $\bigcup_{i=1; i \neq j, j-1}^{R+p-1} [t_i + \delta_i, t_{i+1} - \delta_{i+1}] = S$. Let β_1 be defined by

$$\beta_1 = \min\left\{\min_{x \in S} |\phi(x)|/2 \max_{x \in S} |\Psi(x)|, \alpha\right\}$$

then on S,

$$\beta_1|\Psi(x)| \leqq \left(\min_{x \in S} \frac{|\phi(x)|}{2}\right) \frac{|\Psi(x)|}{\max_{x \in S} |\Psi(x)|} \leqq \min_{x \in S} \frac{|\phi(x)|}{2} < |\phi(x)|.$$

Since it has already been established that

$$|\phi(x)| > \beta|\Psi(x)|$$
$$\text{for } x \in [t_1, t_{R+p}] - [S \cup \{t_1, \ldots, t_{R+p}\} \cup (t_{j-1} + \delta_{j-1}, t_j) \cup (t_j, t_{j+1} - \delta_{j+1})]$$

and $0 < \beta \leqq \beta_1$, it must only be shown for some β in this range, $z_\beta(x) = \phi(x) - \beta\Psi(x)$ has only one zero in (t_{j-1}, t_j) and only one zero in (t_j, t_{j+1}) and these are simple zeros.

Suppose this were not the case. Let t_β be the first zero of z_β in (t_{j-1}, t_j) and $\bar{t}_\beta$ be the last zero of z_β in (t_j, t_{j+1}). z_β has a zero of order m_j-2 at t_j hence by supposition and by Rolle's theorem $z_\beta^{(m_j-2)}$, $z_\beta^{(m_j-1)}$, $z_\beta^{(m_j)}$ all vanish in $(t_\beta, \bar{t}_\beta)$. Also $t_\beta \to t_j$ and $\bar{t}_\beta \to t_j$ as $\beta \to 0$. Thus, letting $\beta \to 0$, $z_0(x)$ has a zero of order m_j+1 at t_j which is impossible since $z_0(x)=\phi(x)$. Hence the desired β exists. But for this β, z_β has a simple zero at a, $R-1$ distinct zeros on $[a, \eta_1(a)]$ and its first n zeros occur at, at least, $R+p+1$ points contradicting the maximality of p.

LEMMA 6. *Let the hypotheses* (i), (ii), (iii) *and* (iv) *of Lemma* 5 *be satisfied. Also let*

(v) m_{R+p} *have minimum value over all solutions of* (1) *satisfying* (i), (ii), (iii) *and* (iv).

Then, if $p \geqq 1$, $m_{R+p}=1$.

Proof. Let $\phi(x)$ be a solution satisfying the hypothesis of the lemma. Suppose the minimum $m_{R+p}>1$. Let $\Psi(x)$ be a solution of (1) with zeros at t_1, $(t_1+t_2)/2$, $t_2, \ldots, t_{R+p-2}, t_{R+p}$ of orders $m_1, 1, m_2, \ldots, m_{R+p-2}, n-(R+p-1)-1<m_{R+p}$ (by (iv)) (remember, by Lemma 5, $m_i=1$, $i=1, 2, \ldots, R+p-1$). By the maximality of R, Ψ can have no other zeros on $[a, b)$. Since $-\phi$ and $-\Psi$ are also solutions, it can be assumed $\phi(x)>0$ and $\Psi(x)>0$ on (t_{R+p-1}, t_{R+p}). Hence

$$\Psi(t_{R+p-1}) > \phi(t_{R+p-1}) = 0, \qquad |\Psi^{(n-R-p)}(t_{R+p})| > \phi^{(n-R-p)}(t_{R+p}) = 0.$$

Thus, $\alpha\Psi$ starts out greater to the left of t_{R+p} than ϕ for any $\alpha>0$. Thus there is an $\alpha>0$ such that for any β, $0<\beta\leqq\alpha$, $\beta\Psi((t_{R+p-1}+t_{R+p})/2)<\phi((t_{R+p-1}+t_{R+p})/2)$ and hence there is an $x_\beta \in (t_{R+p-1}, (t_{R+p-1}+t_{R+p})/2)$ and an $\bar{x}_\beta \in ((t_{R+p-1}+t_{R+p})/2, t_{R+p})$ such that $\phi(x_\beta)=\beta\Psi(x_\beta)$ and $\phi(\bar{x}_\beta)=\beta\Psi(\bar{x}_\beta)$. Note also $\beta_1\neq\beta_2$ implies $x_{\beta_1}\neq x_{\beta_2}$ and $\bar{x}_{\beta_1}\neq\bar{x}_{\beta_2}$. Pick $\alpha_1=\min\{\alpha, |\phi'(t_i)|/2|\Psi'(t_i)| \ (i=1, 2, \ldots, t_{R+p-2})\}$.

On $[t_1, t_2]$, Ψ has one more zero than ϕ and on $[t_2, t_{R+p-2}]$ they have the same number of zeros. Hence ϕ and Ψ have opposite signs on $(t_2, t_{R+p-2})-\{t_3, \ldots, t_{R+p-3}\}$. Also $\phi(x)\neq\beta\Psi(x)$ for $x\in(t_1, t_2)$ since this would imply $z_\beta=\phi-\beta\Psi$ had n zeros and R distinct zeros in $[a, \eta_1(a))$, contrary to the choice of R. It will now be shown that there is an α_2, $\alpha_1>\alpha_2>0$, such that if $0<\beta\leqq\alpha_2$, then z_β has exactly one zero, a simple zero, in $(t_{R+p-1}, (t_{R+p-1}+t_{R+p})/2]$.

If this were not the case then there would exist a sequence $\beta_i\to 0$ as $i\to\infty$ such that if x_{β_i} is the last zero of z_{β_i} in $(t_{R+p-1}, (t_{R+p-1}+t_{R+p})/2]$ then z'_{β_i} vanishes, by Rolle's theorem, in $(t_{R+p-1}, x_{\beta_i}]$. But since $x_{\beta_i}\to t_{R+p-1}$ and $z_{\beta_i}\to\phi$ as $i\to\infty$, this implies ϕ has a double zero at t_{R+p-1} which is impossible by Lemma 5. Hence $\alpha_2>0$ exists.

Now, by Lemma 1, there is a β, $0<\beta<\alpha_2$, for which z_β has for its first zero $\bar{x}_\beta$ in $[t_{R+p-1}, t_{R+p})$ a simple zero. Hence z_β has zeros at $t_1, t_2, \ldots, t_{R+p-2}, x_\beta, \bar{x}_\beta, t_{R+p}$ of orders $m_1, m_2, \ldots, m_{R+p-2}, 1, 1, n-(R+p)$, a total of n zeros at, at least, $R+p$ points. If $n-(R+p)=0$ then z_β has as its $R+p$th zero a simple zero, contradicting the minimality of m_{R+p}. If $n-(R+p)>0$ then z_β has n zeros at, at least, $R+p+1$ points contradicting the maximality of p.

LEMMA 7. *Let the hypothesis* (i), (ii), (iii), (iv), *and* (v) *of Lemma* 6 *be satisfied. Assume also if* $\eta_1((a+\eta_1(a))/2)$ *exists* (*note that* $\eta_1(a)$ *is an increasing function of a by* [3]) *then* $\eta_1(a)<b<\eta_1((a+\eta_1(a))/2)$. *Then* $R=n$, *if* $p=0$.

REMARK The last assumption of this lemma is not a significant restriction since none of the previous results depended on b and $\eta_1(a)<b$ always holds.

Proof. Suppose this is not the case. Then $R<n$ and $m_R \geqq n-(R-1)>1$, and hence $m_R-[(n-R)-1]\geqq 2$. There are two cases.

Case 1. $m_R-[(n-R)-1]$ is even. Let $\Psi(x)$ be a solution of (1) with zeros at $t_1, (t_1+t_2)/2, t_2, \ldots, t_{R-1}, t_R$ of orders $m_1, 1, m_2, \ldots, m_{R-1}, n-R-1<m_R$, a total of $n-1$ zeros and R distinct zeros in $[a, \eta_1(a))$. Hence Ψ has no other zeros on $[a, b)$. It can be assumed $\phi(x)\Psi(x)>0$ for $x \in [t_R-\varepsilon, t_R) \cup (t_R, t_R+\varepsilon]$ for some $\varepsilon>0$ chosen so that $t_{R-1}<t_R-\varepsilon<t_R<t_R+\varepsilon<b$. Also for each β, $0<\beta$, there is an ε_β such that $0<|\phi(x)|<\beta|\Psi(x)|$ for $x \in (t_R-\varepsilon_\beta, t_R) \cup (t_R, t_R+\varepsilon_\beta)$. Pick

$$\alpha = \min \{\phi(t_R-\varepsilon)/2\Psi(t_R-\varepsilon), \phi(t_R+\varepsilon)/2\Psi(t_R+\varepsilon), |\phi'(t_i)|/2|\Psi'(t_i)| \quad (i < R)\}.$$

Then, using the same technique as used in Lemma 5, there is a β_1, $0<\beta_1\leqq\alpha$, such that for $0<\beta\leqq\beta_1$, $z_\beta=\phi-\beta\Psi$ has no zeros in $[t_1, t_R-\varepsilon]$ other than simple zeros at $t_1, \ldots, t_{R-1}$. By Lemma 1 there is a β, $0<\beta\leqq\beta_1$ for which the first zero of z_β in $[t_R-\varepsilon, t_R)$ is a simple zero. Since z_β also vanishes at least once in $(t_R, t_R+\varepsilon)$, z_β has n zeros on $[a, b)$ and its first n zeros at, at least $R+1$ points, contradicting the assumption that $p=0$.

Case 2. $m_R-[(n-R)-1]$ is odd. Let Ψ_1 be a solution of (1) with zeros at $t_1, t_3, \ldots, t_{R-1}, s_1, t_R$ where $t_R\geqq\eta_1(a)>s_1>\max((a+\eta_1(a))/2, t_{R-1})$ of orders $m_1, m_3, \ldots, m_{R-1}, 1, n-R<m_R$, a total of $n-1$ zeros with $R-1$ distinct zeros in $[a, \eta_1(a))$.

If Ψ_1 has no other zeros in $(a, t_R]$ then it can be assumed that $\Psi_1>0$ and $\phi(x)>0$ for $x \in (t_1, t_2)$. Let $\alpha=\min\{|\phi'(t_i)|/2|\Psi_1'(t_i)| \quad (i<R)\}$ (if $\Psi_1'(a)=0$ the term $|\phi'(t_1)|/2|\Psi_1'(t_1)|$ can be omitted). Then, since on $[t_1, t_{R-1}]$, ϕ has one more zero than Ψ_1 and on $[t_{R-1}, t_R)$, ϕ has one less zero than Ψ_1 for any β, $0<\beta\leqq\alpha$, $z_\beta=\phi-\beta\Psi$ has a zero at some point $x_\beta \in (t_1, t_2)$ and a zero at some point $\bar{x}_\beta \in (s_1, t_R)$ in addition to simple zeros at $t_1, t_3, \ldots, t_{R-1}$ and a zero at t_R of order $n-R$, a total of at least n zeros. Further the zero at x_β is simple by Lemma 3 and by Lemma 1, β can be chosen so that the zero at $\bar{x}_\beta$ is a simple zero and z_β has no other zeros in $[a, \bar{x}_\beta)$. Thus z_β has n zeros, a simple zero at a, $R-1$ distinct zeros on $[a, \eta_1(a))$, and its first n zeros occur at, at least, $R+1$ distinct points contradicting $p=0$.

If Ψ_1 has another zero in $(a, t_R]$ it occurs in $[\eta_1(a), t_R]$. First suppose $t_R>\eta_1(a)$ and the additional zero occurs in $[\eta_1(a), t_R)$. Let the first zero of Ψ_1 in (s_1, t_R) be at γ_1, $\eta_1(a)\leqq\gamma_1$. It may be assumed that $\phi\Psi>0$ on (s_1, γ_1). Let $\gamma_2 \in (s_1, \eta_1(a))$ and let $\alpha=\min\{|\Psi'(t_i)|/2|\phi'(t_i)| \ (i<R), \Psi(\gamma_2)/2\phi(\gamma_2)\}$. Then $z_\beta=\Psi-\beta\phi$ for any β, $0<\beta\leqq\alpha$

has simple zeros at $t_1, t_3, \ldots, t_{R-1}$ and (by Lemma 3) at some point $\gamma_\beta \in (s_1, \gamma_2)$ in addition to a zero in (γ_2, γ_1) and a zero of order $n-R$ at t_R, a total of at least n zeros on $[t_1, t_R]$. By (ii) z_β has no other zeros on $[t_1, \gamma_2]$. By Lemma 1, there is a β_1, $0<\beta_1 \leqq \alpha$, such that z_{β_1} has for its first zero in $[\gamma_2, \gamma_1)$ a simple zero. Hence z_{β_1} has its first n zeros occurring at, at least, $R+1$ distinct points, again contradicting $p=0$.

Now suppose $t_R=\eta_1(a)$ and Ψ_1 has an additional zero at t_R. Hence Ψ_1 and ϕ have $n-1$ zeros in common and $R-2$ common zeros on $[a, \eta_1(a))$. It can be assumed that $\phi\Psi>0$ on (s_1, t_R). Let $\bar{s}_1 \in (s_1, \eta_1(a))$ and let $z_1(x)=\phi(x) -(\phi(\bar{s}_1)/\Psi_1(\bar{s}_1))\Psi_1(x)$ where $\bar{s}_1$ is chosen so that $\phi'(t_1)/\Psi_1'(t_1) \neq \phi(\bar{s}_1)/\Psi_1(\bar{s}_1)$ (if $\Psi_1'(t_1)=0$, this is of course a superficial requirement). Note, this ratio is not constant on $(s_1, \eta_1(a))$ since ϕ and Ψ_1 are linearly independent by the way which Ψ_1 was chosen. $z_1(x)$ has n zeros on $[a, t_R]$, a simple zero at a, and $R-2$ zeros on $(a, \eta_1(a))$ which by Lemma 3 are simple zeros. Repeat the above procedure, starting with case 1, with z_1 taking the place of ϕ and Ψ_1 being replaced by Ψ_2, a solution with zeros at $t_1, t_4, \ldots, t_{R-1}, \bar{s}_1, s_2, t_R$, where $\eta_1(a)>s_2>\bar{s}_1$, of orders $m_1, m_4, \ldots, m_{R-1}, 1, 1, n-R<m_R$. After at most $R-2$ steps we have a z_{R-2}, with zeros at $t_1, \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_{R-2}, t_R$ of orders $m_1, 1, 1, \ldots, 1, n-R+1$, which cannot have another zero at t_R since $(a+\eta_1(a))/2<\bar{s}_1<\bar{s}_2<\cdots<\bar{s}_{R-2}<t_R<\eta_1(a+\eta_1(a))/2$. Hence the zero of z_{R-2} at t_R is of order exactly $n-R+1$. However $m_R-[(n-R)-1] \geqq 2$ and is odd hence $m_R-[(n-R)-1]\geqq 3$ thus $m_R-[n-R+1]\geqq 1$ contradicting the minimality of m_R.

LEMMA 8. *Let the hypothesis of Lemma 7 be satisfied. Then if $p=0$, $m_R=1$.*

Proof. From Lemma 7, $R=n$. Let $\phi(x)$ be a solution of (1) with zeros at $a=t_1<t_2<\cdots<t_R$ of orders $m_1=1, \ldots, m_{R-1}=1, m_R$. Suppose $m_R>1$. Since $\eta_1(a)$ is an increasing function of a, $\eta_1(t_2)>\eta_1(a)$, hence $\phi(\eta_1(a))\neq 0$ and $t_R>\eta_1(a)$. Let $\Psi_1(x)$ be a solution of (1) with zeros at $t_1, t_3, \ldots, t_{n-1}, s_1$, where $\eta_1(a)>s_1 >\max((a+\eta_1(a))/2, t_{n-1})$ of orders $m_1=1, m_3=1, \ldots, m_{n-1}=1, 1$, a total of $n-1=R-1$ zeros, all distinct on $[a, \eta_1(a))$. Clearly Ψ_1 has no more zeros in $[a, \eta_1(a))$. Since $\eta_1(a)$ is an increasing function of a, if $\Psi_1(\eta_1(a))=0$ it is a simple zero and the proof is complete. Similarly if Ψ_1 vanishes in $(\eta_1(a), b)$ and its first zero in that interval is simple, the proof is completed. Hence either Ψ_1 vanishes in $(\eta_1(a), b)$ with a multiple zero as its first zero in that interval or Ψ_1 does not vanish in $(\eta_1(a), b)$.

Suppose first Ψ_1' does not vanish in $(\eta_1(a), b)$. It can be assumed $\phi(x)\Psi_1(x)>0$ on $(t_1, t_2) \cup (s_1, t_n)$. Then since $|\Psi_1(t_n)|>|\phi(t_n)|=0$ and $|\phi(s_1)|>|\Psi_1(s_1)|=0$, for any $\alpha>0$, $\alpha\Psi_1(x)=\phi(x)$ for some $x_\alpha \in (s_1, t_n)$. Further if α is chosen so that $\alpha<\phi'(t_1)/\Psi_1'(t_1)$ (note $\Psi_1'(t_1)\neq 0$) then $\alpha\Psi_1(x)=\phi(x)$ for some $\bar{x}_\alpha \in (t_1, t_2)$. This follows since for α in this range $|\phi|$ starts out larger to the right of t_1 than $|\Psi_1|$, but $|\phi|$ vanishes (at the point t_2) before $|\Psi_1|$. Then $z_\alpha(x)=\phi(x)-\alpha\Psi_1(x)$ has zeros at $t_1, \bar{x}_\alpha, t_3, \ldots, t_{n-1}$, a total of $n-1$ zeros in $[a, \eta_1(a)]$. Hence z_α has no other zeros in $[a, \eta_1(a)]$. Clearly z_β, $0<\beta\leqq\alpha$ has the same properties. Lemma 1 can now be applied to show there is a β in this range for which z_β has as its first zero in $[\eta_1(a), t_n)$ a simple zero, completing the proof.

Suppose now Ψ_1 does vanish in $(\eta_1(a), b)$. Repeat the above proof of this lemma with ϕ replaced by Ψ_1, and Ψ_1 replaced by Ψ_2, a solution with zeros at $t_1, t_4, \ldots, s_1, s_2$, where $s_1<s_2<\eta_1(a)$. After at most $n-2$ repetitions a solution Ψ_{n-2} is found with zeros at $t_1, s_1, s_2, \ldots, s_{n-2}$ where $t_1<(a+\eta_1(a))/2<s_1<s_2<\cdots<s_{n-2}<\eta_1(a)$. Ψ_{n-2} cannot have a multiple zero in $(\eta_1(a), b)$ since if it did Ψ_{n-2} would have n zeros in $((a+\eta_1(a))/2, \eta_1((a+\eta_1(a))/2))$ which is impossible. Hence repeating the proof for this solution leads to the desired conclusion.

THEOREM 1. *Suppose there is a solution of* (1) *with a zero at a and n zeros on* $[a, b)$. *Then there is a solution with a simple zero at a whose first n zeros on* $[a, b)$ *are simple zeros.*

Proof. If $\eta_1((a+\eta_1(a))/2)$ exists let $b_1\in(\eta_1(a), b)\cap(\eta_1(a), \eta_1((a+\eta_1(a))/2))$. The remainder of the proof consists of reading the statements of Lemmas 2, 3, 4, 5, 6, 7, and 8 with b replaced by b_1.

3. **Continuity of** $\eta_1(a)$. Using Theorem 1, it shall be shown that $\eta_1(a)$ is a continuous function of a and the coefficients in (1). First a lemma which follows from the results of [5] will be established.

LEMMA 9. *Let a be any point for which* $\eta_1(a)$ *exists; then for any* $b>\eta_1(a)$, *there is a* $c>a$ *such that* $\eta_1(c)=b$.

Proof. This follows from Theorem 6 of [5] after the change of variables $t=-x$.

THEOREM 2. $\eta_1(a)$ *is a continuous function of a.*

Proof. Let $\varepsilon>0$. By Theorem 1 there is a solution ϕ of (1) with a simple zero at a whose first n zeros on $[a, \eta_1(a)+\varepsilon)$ are simple zeros. Let these first n zeros be at $a_1=t_1<t_2<\cdots<t_n$ $(<\eta_1(a)+\varepsilon)$. Let $t_{n+1}=\min\{\eta_1(a)+\varepsilon/2, \min b>t_n$ such that $\phi(b)=0\}$. Since a constant times a solution is again a solution it can be supposed that $\varepsilon<\min\{\max_{x\in[t_i,t_{i+1}]}|\phi(x)|, (i=1, 2, \ldots, n)\}$. Since solutions are continuous functions of these initial conditions there is a $\delta_1>0$ such that if $a_1\in(a-\delta_1, a+\delta_1)$ and Ψ is a solution of (1) satisfying $\Psi^{(i)}(a_1)=\phi^{(i)}a$, $i=0, 1, \ldots, n-1$, then $|\phi(x)-\Psi(x)|<\varepsilon$ for $x\in[a-\varepsilon, \eta_1(a)+\varepsilon]$ (see [4, p. 56]). It follows that Ψ must vanish in each of the intervals $(t_i-\varepsilon, t_i+\varepsilon)$, $i=1, 2, \ldots, n$. Hence Ψ has n zeros on $[a_1, \eta_1(a)+\varepsilon)$ and hence $\eta_1(a_1)<\eta_1(a)+\varepsilon$. If $a_1>a$ then since $\eta_1(a)$ is an increasing function of a, $\eta_1(a_1)\in(\eta_1(a), \eta_1(a)+\varepsilon)$. If $a_1<a$ then $\eta_1(a_1)<\eta_1(a)$. If $\eta_1(a_1)>\eta_1(a)-\varepsilon$ pick $\delta=|a-a_1|$. If $\eta_1(a_1)\leqq\eta_1(a)-\varepsilon$ then by Lemma 9, $\eta_1(a)-\varepsilon/2=\eta_1(c)$ for some $c\in(a_1, a)$ in which case let $\delta=|a-c|<|a-a_1|$. Hence for any x such that $|x-a|<\delta$, $|\eta_1(a)-\eta_1(x)|<\varepsilon$, which completes the proof (for another proof of this result see [3]).

THEOREM 3. *$\eta_1(a)$ is a continuous function of the coefficients $p_i(x)$, $i=0, 1, \ldots, n-1$.*

Proof. Let $\varepsilon>0$. Let ϕ be a solution of (1) with a simple zero at a whose first n zeros on $[a, \eta_1(a)+\varepsilon)$ are simple zeros. Let these first n zeros of ϕ be at $a=t_1<t_2<\cdots<t_n$. Let $t_{n+1}=\min\{\eta_1(a)+\varepsilon/2, \min b>t_n$ such that $\phi(b)=0\}$. Since a constant times a solution is again a solution it can be supposed that

$$\varepsilon < \min\left\{\max_{x\in[t_i,t_{i+1}]} |\phi(x)| \quad (i = 1, 2, \ldots, n)\right\}.$$

Since solutions are continuous functions of their coefficients, there is a $\delta>0$ such that if $|q_i(x)-p_i(x)|<\delta$ for $x\in[a, \eta_1(a)+\varepsilon]$ and $i=0, 1, \ldots, n-1$, and $y_1(x)$ is a solution of (1) and $z_1(x)$ is a solution of

$$z^{(n)} + \sum_{i=0}^{n-1} q_i(x)z^{(i)} = 0 \tag{2}$$

satisfying $y_1^{(i)}(a)=z_1^{(i)}(a)$, $i=0, 1, \ldots, n-1$, then $|y_1(x)-z_1(x)|<\varepsilon$ for $x\in[a, \eta_1(a)+\varepsilon]$. Let $\Psi(x)$ be the solution of (2) satisfying $\Psi^{(i)}(a)=\phi^{(i)}(a)$, $i=0, 1, \ldots, n-1$. Then $\Psi(x)$ has n zeros on $[a, \eta_1(a)+\varepsilon)$. Hence if $\bar\eta_1(a)$ denotes the first conjugate point of a relative to (2), $a<\bar\eta_1(a)<\eta_1(a)+\varepsilon$. To complete the proof it needs merely to be shown that $\bar\eta_1(a)>\eta_1(a)-\varepsilon$.

Suppose this were not the case. Then $\bar\eta_1(a)\leqq\eta_1(a)-\varepsilon$. Then, by Theorem 1 again, there is a solution $\Psi_1(x)$ of (2) with a simple zero at a whose first n zeros on $[a, \bar\eta_1(a)+\varepsilon/2)$ are simple zeros. Let these zeros be at $a=\bar t_1<\bar t_2<\cdots<\bar t_n$. Let $\bar t_{n+1}=\min\{\eta_1(a)+\varepsilon/2, \min b>\bar t_n$ such that $\Psi_1(b)=0\}$. Again it may be supposed that $\varepsilon<\min\{\max_{x\in[\bar t_i,\bar t_{i+1}]} |\Psi_1(x)|\ (i=1, 2, \ldots, n)\}$. Hence, by the continuity of solutions with respect to the coefficients, the solution $\phi_1(x)$ of (1) satisfying the initial conditions $\phi^{(i)}(a)=\Psi_1^{(i)}(a)$ has n zeros on $[a, \bar\eta_1(a)+\varepsilon/2]$. This, however, is impossible since $\bar\eta_1(a)+\varepsilon/2<\eta_1(a)$. The proof of the theorem is thus complete.

4. **Open, closed and half open intervals.** It shall be shown by example that the interval $[a, b)$ in Theorem 1 cannot in general be replaced by the closed interval $[a, b]$.

In fact, consider the equation $y'''+y=0$. $\eta_1(0)$ for this equation is achieved by a solution ϕ with a double zero at 0 and a simple zero at $\eta_1(0)$. Further it follows from Theorem 2.5 of [1] that no nontrivial solution with a double zero at $\eta_1(0)$ can vanish in $[0, \eta_1(0))$. If there were a solution Ψ with three distinct zeros on $[0, \eta_1(0)]$ then it would have to vanish at 0 and have a simple zero at $\eta_1(0)$. Hence there would exist a solution which is a linear combination of ϕ and Ψ with a zero at 0 and a double zero at $\eta_1(0)$, which is impossible. A further discussion of equations of this type can be found in [1]. The existence of the solution in this example is an application of

THEOREM 4. *Suppose there is a solution of* (1) *with n zeros on $[a, \eta_1(a)]$, and at least α zeros in $[a, \eta_1(a))$, $((a, \eta_1(a)])$. Then there is a solution of* (1) *with a zero of order at least α at a $(\eta_1(a))$ and n zeros on $[a, \eta_1(a)]$.*

Proof. Let $s_1=\{\phi \mid \phi$ is a solution of (1) with n zeros on $[a, \eta_1(a)]$ and at least α zeros in $[a, \eta_1(a))\}$. Let $s_2=\{k \mid$ there is a $\phi \in s_1$ with a zero of order k at $x=a\}$. Let $m=\max k \in s_2$. Let $s_3=\{k \mid$ there is a $\phi \in s_1$ with zero of order m at a and k zeros in $(a, \eta_1(a))\}$. Let $M=\max k \in s_3$. Let ϕ be a solution of (1) with zeros at $a=t_1<t_2<\cdots<t_q=\eta_1(a)$ of orders $m=m_1, m_2, \ldots, m_q$ respectively where $\alpha \leqq \sum_{i<q} m_i \leqq n-1<\sum_{i\leqq q} m_i$ and $M=\sum_{1<i<q} m_i$.

Suppose first that $m_1<\alpha$. Then $q>2$ since $\phi \in s_1$. Let Ψ be a solution of (1) with zeros at $a=t_1, t_2, \ldots, t_q=\eta_1(a)$ of orders $m_1+1, m_2, \ldots, m_{q-1}-1, n-\sum_{i<q} m_i-1$, a total of $n-1$ zeros in $[a, \eta_1(a)]$ and a total of $\sum_{i<q} m_i \geqq \alpha$ zeros in $[a, \eta_1(a))$. By the maximality of m_1, Ψ can have no other zeros in $[a, \eta_1(a)]$. By Theorem 1 of [3] there is a linear combination Ψ_1 of Ψ and ϕ with a double zero at some point $u_1 \in (t_{q-1}, t_q)$. Hence Ψ_1 has zeros at $a=t_1, \ldots, t_{q-1}, u_1, t_q$ of orders $m_1, \ldots, m_{q-1}-1, 2, n-\sum_{i<q} m_i-1$. Thus $\Psi_1 \in s_1$ with a zero at a of order m_1, and a total of $M+1$ zeros in $(a, \eta_1(a))$, contradicting the maximality of M. Hence $m_1 \geqq \alpha$.

The connection between open, closed and half open intervals with regard to Theorem 1 is shown in the following theorem.

THEOREM 5. *The following statements are equivalent*:

(a) *There is a solution of* (1) *with n zeros on* (a, b).

(b) *There is a solution of* (1) *with n zeros on* $[a, b)$.

(c) *There is a largest point* $a_1 \in (a, b)$ *such that for any* $c \in [a, a_1)$, $\eta_1(c)<b$ *and for this point* a_1, $\eta_1(a_1) \notin (a, b)$.

(d) *There is a largest point* $a_2 \in (a, b)$ *such that for any* $c \in [a, a_2)$ *there is a solution of* (1) *with a simple zero at c whose first n zeros on* $[c, b)$ *are simple zeros.*

Proof. (a) $\Rightarrow$ (b) is trivial. (b) $\Rightarrow$ (c) follows since $\eta_1(a)$ is a continuously increasing function of a (Theorem 2 above and Theorem 7 of [5]). (c) $\Rightarrow$ (d) follows from Theorem 1. (d) $\Rightarrow$ (a) is trivial.

5. **Solutions with n simple zeros on** $[a, \eta_1(a)]$. The results in this section give a more complete picture of the situation when there is a solution with n distinct zeros on $[a, \eta_1(a)]$. It is trivial to show that if there is a solution with n distinct zeros on $[a, \eta_1(a)]$ then these zeros are simple zeros.

THEOREM 6. *Suppose there is a solution* $\phi_1(x)$ *of* (1) *with n simple zeros on* $[a, \eta_1(a)]$. *Then there exist solutions* $\phi_2, \phi_3, \ldots, \phi_{n-1}$ *such that* ϕ_k *has n zeros on* $[a, \eta_1(a)]$, *a zero at a of order k*, $n-k-1$ *simple zeros in* $(a, \eta_1(a))$ *and no other zeros in* $[a, \eta_1(a))$. *Further the zeros of* $\phi_{k+1}(x)$ *in* $(a, \eta_1(a))$ *coincide with the first* $n-k-2$ *zeros of* $\phi_k(x)$ *in* (a, b).

Proof. Let the zeros of ϕ_1 be at $a=t_1<t_2<\cdots<t_n=\eta_1(a)$ all of order one. Let ϕ_2 be a solution of (1) with a zero at t_1 of order 2, and simple zeros at $t_2, t_3, \ldots, t_{n-2}$, a total of $n-1$ zeros. Hence ϕ_2 has no other zeros on $[a, \eta_1(a))$. If ϕ_2 did not vanish at $\eta_1(a)$ then, by Theorem 1 of [5], there is a linear combination of ϕ_1 and ϕ_2 with

a double zero in (t_{n-1}, t_n) and simple zeros at $t_1, \ldots, t_{n-2}$, a total of n zeros in $[a, \eta_1(a))$ which is impossible. Hence ϕ_2 vanishes at $\eta_1(a)$. Suppose the existence of ϕ_k has been established then define ϕ_{k+1} to have a zero at a of order $k+1$, and have simple zeros coincide with the first $n-k-2$ simple zeros of ϕ_k in $(a, \eta_1(a))$, a total of $n-1$ zeros. Again ϕ_{k+1} can have no other zeros in $[a, \eta_1(a))$ and if ϕ_{k+1} did not vanish at $\eta_1(a)$ there would be a linear combination of ϕ_k and ϕ_{k+1} with n zeros in $[a, \eta_1(a))$, which is impossible. Hence the existence of ϕ_{k+1} is established and the theorem follows by induction.

COROLLARY. *Under the hypothesis of Theorem* 6 *there exist solutions* $\phi_2, \phi_3, \ldots, \phi_{n-1}$ *such that* ϕ_k *has n zeros on* $[a, \eta_1(a)]$, *a zero at* $\eta_1(a)$ *of order* k, $n-k-1$ *simple zeros in* $(a, \eta_1(a))$ *and no other zeros in* $(a, \eta_1(a)]$.

LEMMA 10. *Suppose there is a solution* $\phi_1(x)$ *of* (1) *with n simple zeros on* $[a, \eta_1(a)]$. *Let* $\phi_2, \phi_3, \ldots, \phi_{n-1}$ *be the solutions obtained in Theorem* 6. *Let the zeros of* $\phi_1, \ldots, \phi_{n-1}$ *at* $\eta_1(a)$ *be of orders* $n_1 = 1, n_2, \ldots, n_{n-1}$. *Then* $n_1 \leqq n_2 \leqq \cdots \leqq n_{n-1}$.

Proof. Suppose this were not the case. Then there is an index j such that $n_j > n_{j+1}$. Letting $t_1, \ldots, t_n$ be as in the proof of Theorem 6 it follows that $\phi_j(t_{n-j}) = 0$ and $\phi_{j+1}(x) \neq 0$ for $t_{n-j} \leqq x < t_n$. Thus, by Theorem 1 of [5], there is a linear combination of ϕ_j and ϕ_{j+1} with a double zero in (t_{n-j}, t_n), simple zeros at $t_2, \ldots, t_{n-j-1}$, and a zero at a of order j. A total of $2+n-j-2+j=n$ zeros in $[a, \eta_1(a))$ which is a contradiction to the definition of $\eta_1(a)$.

THEOREM 7. *Suppose there is a solution of* (1) *with n simple zeros on* $[a, \eta_1(a)]$. *Then there exists solutions* $\Psi_1, \Psi_2, \ldots, \Psi_{n-1}$, *not necessarily distinct, such that* Ψ_k *has a zero at a of order at least* $n-k$ *and a zero at* $\eta_1(a)$ *of order at least* k.

Proof. Let $\phi_1, \phi_2, \ldots, \phi_{n-1}$ be the solution obtained in Theorem 6. Define $\Psi_{kj}(x)$ $(k=1, 2, \ldots, n-1; j=k, k+1, \ldots, n-1)$ by

$$\begin{aligned}
\Psi_{11}(x) &= \phi_{n-1}(x); \\
\Psi_{1j}(x) &= \phi'_{n-j}(\eta_1(a))\phi_{n-j+1}(x) - \phi'_{n-j+1}(\eta_1(a))\phi_{n-j}(x), \quad \text{if } \phi'_{n-j+1}(\eta_1(a)) \neq 0, \\
&= \phi_{n-j+1}(x), \quad \text{if } \phi'_{n-j+1}(\eta_1(a)) = 0; \\
\Psi_{kk}(x) &= \Psi_{k-1\,k}(x); \\
\Psi_{kj}(x) &= \Psi^{(k)}_{k-1\,j}(\eta_1(a))\Psi_{k-1\,j-1}(x) - \Psi^{(k)}_{k-1\,j-1}(\eta_1(a))\Psi_{k-1\,j}(x), \\
&\qquad\qquad \text{if } \Psi^{(k)}_{k-1\,j}(\eta_1(a)) \neq 0 \neq \Psi^{(k)}_{k-1\,j-1}(\eta_1(a)), \\
&= \Psi_{k-1\,j}(x), \quad \text{if } \Psi^{(k)}_{k-1\,j}(\eta_1(a)) = 0 \neq \Psi^{(k)}_{k-1\,j-1}(\eta_1(a)), \\
&= \Psi_{k-1\,j-1}(x), \quad \text{if } \Psi^{(k)}_{k-1\,j-1}(\eta_1(a)) = 0.
\end{aligned}$$

The claim is now that $\Psi_k = \Psi_{kk}$. To establish this some properties of the Ψ_{kj} will first be established.

(i) Ψ_{kj} has a zero at $\eta_1(a)$ of order at least $k+1$ for $j > k$ and of order at least k if $j = k$,

(ii) Ψ_{kj} has a zero at a of order at least $n-j$.

These properties will be established by induction on k.

First let $k=1$. For $j=1$ it is obvious. For $j>1$ the result is also obvious if $\phi'_{n-j+1}(\eta_1(a))=0$. If $\phi'_{n-j+1}(\eta_1(a))\neq 0$ then, by Lemma 9, $\phi'_{n-j}(\eta_1(a))\neq 0$ and hence $\Psi'_{1j}(\eta_1(a))=0$ and Ψ_{1j} has a zero of the same order as ϕ_{n-j} at a, namely $n-j$. This establishes (i) and (ii) for $k=1$.

Now suppose these properties hold for $k=p$. Then for $k=p+1$ and $j=p+1$, $\Psi_{p+1\ p+1}(x)=\Psi_{p\ p+1}(x)$ hence $\Psi_{p+1\ p+1}(x)$ has a zero at $\eta_1(a)$ of order at least $p+1$ and a zero at a of order at least $n-(p+1)$. Hence $\Psi_{p+1\ p+1}$ has the desired properties. Suppose $k=p+1$ and $j>p+1$. If $\Psi^{(p+1)}_{p\,j-1}(\eta_1(a))=0$ then $\Psi_{p\ j-1}$ has a zero at $\eta_1(a)$ of order at least $p+2$ and a zero at a of order at least $n-(j-1)>n-j$. Hence the properties hold in this case. If $\Psi^{(p+1)}_{pj}(\eta_1(a))=0\neq\Psi^{(p+1)}_{p\,j-1}(\eta_1(a))$ then $\Psi_{p\ j-1}$ has a zero at $\eta_1(a)$ of order at least $p+2$ and a zero at a of order at least $n-j$. Hence the properties hold in this case also. If $\Psi^{(p+1)}_{pj}(\eta_1(a))\neq 0\neq\Psi^{(p+1)}_{p\,j-1}(\eta_1(a))$ then $\Psi_{p+1\ j}$ obviously has a zero at $\eta_1(a)$ of order at least $p+2$ since Ψ_{pj} and $\Psi_{p\ j-1}$ already have zeros at $\eta_1(a)$ of order $p+1$. Also $\Psi_{p+1\ j}$ has a zero at a of order at least $n-j$ since both $\Psi_{p\ j-1}$ and Ψ_{pj} have this many zeros at a. This (i) and (ii) hold in all cases.

These properties give the desired result.

By way of partial converse to the results, the following theorem will be established:

THEOREM 8. *Suppose there are $n-1$ linearly independent solutions of* (1), $\Psi_1,\dots,\Psi_{n-1}$ *such that Ψ_k has a zero at a of order exactly $n-k$, a zero at $\eta_1(a)$ of order exactly k and Ψ_k does not vanish in $(a,\eta_1(a))$. Then there are $n-1$ solutions $\phi_1,\dots,\phi_{n-1}$ such that ϕ_k has a zero at a of order exactly $[(n-k+2)/2]$, a zero at $\eta_1(a)$ of order exactly $[(n-k+1)/2]$ and exactly $k-1$ distinct zeros in $(a,\eta_1(a))$, where $[x]$ denotes the largest integer not exceeding x. In particular there is a solution, namely ϕ_{n-1}, with n simple zeros on $[a,\eta_1(a)]$.*

Proof. It will first be established by induction that a sequence $\phi_1,\dots,\phi_{n-1}$ of solutions exists such that ϕ_k has a zero at a of order exactly $[(n-k+2)/2]$, a zero at $\eta_1(a)$ of order exactly $[(n-k+1)/2]$, at least $k-1$ distinct zeros in $(a,\eta_1(a))$ and ϕ_{k+1} has at least one more distinct zero in $(a,\eta_1(a))$ than ϕ_k. The proof will be by induction on k.

If n is even, then $[(n-1+2)/2]=n/2$ and $[(n-1+1)/2]=n/2$, hence ϕ_1 is the solution $\Psi_{n/2}$ which exists by hypothesis. If n is odd, then $[(n-1+2)/2]=(n+1)/2$ and $[(n-1+1)/2]=(n-1)/2$, hence ϕ_1 is the solution $\Psi_{(n+1)/2}$. Suppose the result is now established for $k=p$.

Suppose first $[(n-p+2)/2]=(n-p+1)/2$. Then ϕ_p has a zero at a of order $(n-p+1)/2$ and a zero at $\eta_1(a)$ of order $(n-p+1)/2$ and $q-1\geqq p-1$ distinct zeros in $(a,\eta_1(a))$. Let the zeros of ϕ_p be at $a=t_0<t_1<\cdots<t_q=\eta_1(a)$. Then on (t_j,t_{j+1}), $j=0,\dots,q-1$, $|\phi_p|>0$. Let $m_j=\max_{x\in[t_j,t_{j+1}]}|\phi_p(x)|$, $0\leqq j\leqq q-1$, and let $m=\min_{0\leqq j\leqq q-1}m_j$. ϕ_p has the same sign on at least $[(q+1)/2]$ of the subintervals

$(t_j, t_{j+1}), j=0, \ldots, q-1$. Since $-\phi_p$ is also a solution it may be assumed that ϕ_p is positive on at least $[(q+1)/2]$ of these subintervals. If q is even and ϕ_p is positive on exactly $q/2$ of the subintervals, take $\phi_p > 0$ on (t_{q-1}, t_q). It can also be assumed that $\Psi_{(n-p-1)/2}$ is positive on $(a, \eta_1(a))$. Let $M = \max_{x\in[a,\eta_1(a)]} \Psi_{(n-p-1)/2}(x)$. Then $(m/M)\Psi_{(n-p-1)/2}$ intersects ϕ_p at least $2[(q+1)/2] \geqq q$ distinct times on $(a, \eta_1(a))$. Hence the function $\phi_p - (m/M)\Psi_{(n-p-1)/2} = \phi_{p+1}$ has at least $q \geqq p$ distinct zeros on $(a, \eta_1(a))$, a zero at a of order

$$\min\left(\left[\frac{n-p+2}{2}\right], n-\frac{n-p-1}{2}\right) = \frac{n-p+1}{2} = \left[\frac{n-(p+1)+2}{2}\right],$$

and a zero at $\eta_1(a)$ of order

$$\min\left(\left[\frac{n-p+1}{2}\right], \frac{n-p-1}{2}\right) = \frac{n-p-1}{2} = \left[\frac{n-(p+1)+1}{2}\right].$$

Suppose now $[(n-p+2)/2] = (n-p+2)/2$. Then ϕ_p has a zero at a of order $(n-p+2)/2$, a zero at $\eta_1(a)$ of order $(n-p)/2$ and at least $q-1 \geqq p-1$ distinct zeros in $(a, \eta_1(a))$. Let the t_p $(0 \leqq j \leqq q-1)$ and the m be as chosen above. ϕ_p can be assumed positive on at least $[(q+1)/2]$ of the subintervals $(t_j, t_{j+1}), j=0, \ldots, q-1$. If q is even and ϕ_p is positive on exactly $q/2$ of the subintervals, take $\phi_p > 0$ on (t_0, t_1). $\Psi_{(n+p)/2}$ can be assumed positive on $(a, \eta_1(a))$. Let $M = \max_{x\in[a,\eta_1(a)]} \Psi_{(n+p)/2}$. Then as in the previous case $\phi_{p+1} = \phi_p - (m/M)\Psi_{(n+p)/2}$.

To complete the proof of the theorem it need only be shown that $q > p$ is impossible. If $q > p$ then since the number of zeros of ϕ_k in $(a, \eta_1(a))$ is a strictly increasing function of k, this would imply ϕ_{n-1} has at least $n-1$ distinct zeros on $(a, \eta_1(a))$ and hence n zeros on $[a, \eta_1(a))$ which is impossible and the proof is complete.

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Arizona State University,
Tempe, Arizona